

MATH 303 — Measures and Integration
Lecture Notes, Fall 2024

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CHAPTER 1

Motivating Problems of Measure Theory

1. The Problem of Measurement

A basic (and very old) problem in mathematics is to compute the size (length, area, volume) of geometric objects. Areas of polygons and circles can be computed by elementary methods. More complicated regions bounded by continuous curves can be attacked with methods from calculus. But what about more general subsets of Euclidean space? Does it always make sense to talk about the (hyper-)volume of a subset of $\mathbb{R}^d$? What properties does volume have, and how do we compute it?

We will consider these general questions as the “problem of measurement” in Euclidean space and discuss some approaches to a solution.

2. Riemann Integration and Jordan Content

A good first attempt at solving the problem of measurement comes from the Riemann theory of integration. The basic strategy is to approximate general regions by finite collections of boxes (sets of the form $B = \prod_{i=1}^d [a_i, b_i]$). For such a box B , we declare the volume to be $\text{Vol}(B) = \prod_{i=1}^d (b_i - a_i)$ and use this to define the volume of more general regions. We will now make this idea rigorous.

DEFINITION 1.1: DARBOUX INTEGRATION

Let $B = \prod_{i=1}^d [a_i, b_i]$ be a box in $\mathbb{R}^d$, and let $f : B \rightarrow \mathbb{R}$ be a bounded function.

- A *Darboux partition* of B is a family of finite sequences $(x_{i,j})_{1 \leq i \leq d, 0 \leq j \leq n_i}$ such that $a_i = x_{i,0} < x_{i,1} < \cdots < x_{i,n_i} = b_i$ for each $i \in \{1, \dots, d\}$.

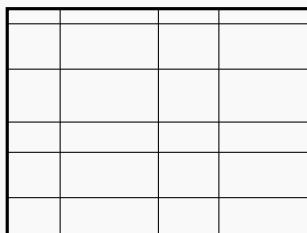


FIGURE 1.1. A Darboux partition in dimension $d = 2$ with $n_1 = 4$ and $n_2 = 6$.

- Given a Darboux partition $P = (x_{i,j})_{1 \leq i \leq d, 0 \leq j \leq n_i}$ of B , the *upper* and *lower Darboux sums of f over B* are given by

$$U_B(f, P) = \sum_{\mathbf{j} \in \prod_{i=1}^d \{1, \dots, n_i\}} \sup_{\mathbf{x} \in B_{\mathbf{j}}} f(\mathbf{x}) \cdot \text{Vol}(B_{\mathbf{j}})$$

and

$$L_B(f, P) = \sum_{\mathbf{j} \in \prod_{i=1}^d \{1, \dots, n_i\}} \inf_{\mathbf{x} \in B_{\mathbf{j}}} f(\mathbf{x}) \cdot \text{Vol}(B_{\mathbf{j}}),$$

where $B_{\mathbf{j}}$ is the box $\prod_{i=1}^d [x_{i,j_i-1}, x_{i,j_i}]$, and $\text{Vol}(B_{\mathbf{j}}) = \prod_{i=1}^d (x_{i,j_i} - x_{i,j_i-1})$ is the volume of $B_{\mathbf{j}}$.

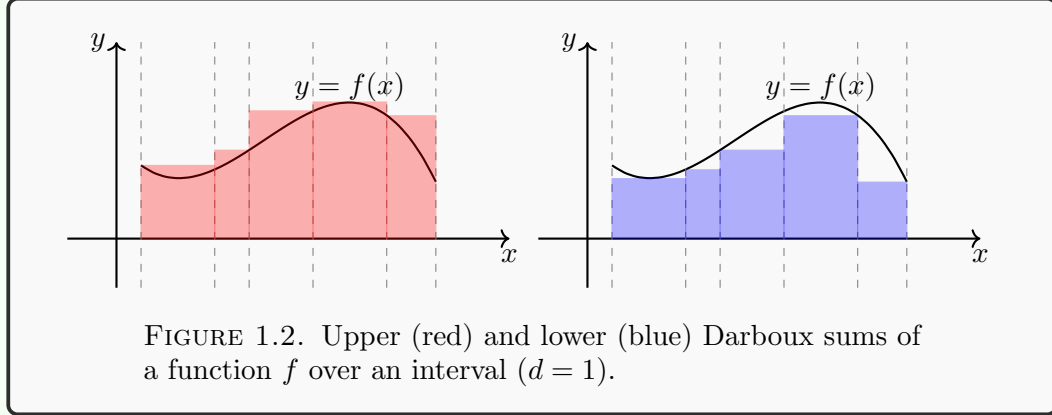


FIGURE 1.2. Upper (red) and lower (blue) Darboux sums of a function f over an interval ($d = 1$).

- The *upper* and *lower Darboux integral of f over B* are

$$U_B(f) = \inf\{U_B(f, P) : P \text{ is a Darboux partition of } B\}$$

and

$$L_B(f) = \sup\{L_B(f, P) : P \text{ is a Darboux partition of } B\}.$$

- The function f is *Darboux integrable over B* if $U_B(f) = L_B(f)$, and their common value is called the *Darboux integral of f over B* and is denoted by $\int_B f(\mathbf{x}) d\mathbf{x}$.

PROPOSITION 1.2

A function f is Darboux integrable if and only if it is Riemann integrable. Moreover, the value of the Darboux integral and the Riemann integral (for a Riemann–Darboux integrable function) are the same.

DEFINITION 1.3

A bounded set $E \subseteq \mathbb{R}^d$ is a *Jordan measurable set* if $\mathbb{1}_E$ is Riemann–Darboux integrable over a box containing E . The *Jordan content* of a Jordan measurable set E is the value $J(E) = \int_B \mathbb{1}_E(\mathbf{x}) d\mathbf{x}$, where B is any box containing E .

Jordan measurable sets include basic geometric objects such as polyhedra, conic sections, regions bounded by finitely many smooth curves/surfaces, etc.

DEFINITION 1.4

A set $S \subseteq \mathbb{R}^d$ is a *simple set* if it is a finite union of boxes $S = \bigcup_{j=1}^k B_j$.

If the boxes $B_1, \dots, B_k$ are disjoint, then the volume of the simple set $S = \bigcup_{j=1}^k B_j$ is $\text{Vol}(S) = \sum_{j=1}^k \text{Vol}(B_j)$. If some of the boxes intersect, then the volume of $S = \bigcup_{j=1}^k B_j$ can be computed using inclusion-exclusion:

$$\text{Vol}(S) = \sum_{j=1}^k \text{Vol}(B_j) - \sum_{1 \leq j_1 < j_2 \leq k} \text{Vol}(B_{j_1} \cap B_{j_2}) + \sum_{1 \leq j_1 < j_2 < j_3 \leq k} \text{Vol}(B_{j_1} \cap B_{j_2} \cap B_{j_3}) - \dots$$

This expression is well-defined, since the intersection of two boxes is again a box. A Jordan measurable set is a set that is “well-approximated” by simple sets, as we will make precise now.

DEFINITION 1.5

For a bounded set $E \subseteq \mathbb{R}^d$, define the *inner* and *outer Jordan content* by

$$J_*(E) = \sup \{ \text{Vol}(S) : S \subseteq E \text{ is a simple set} \}.$$

and

$$J^*(E) = \inf \{ \text{Vol}(S) : S \supseteq E \text{ is a simple set} \}.$$

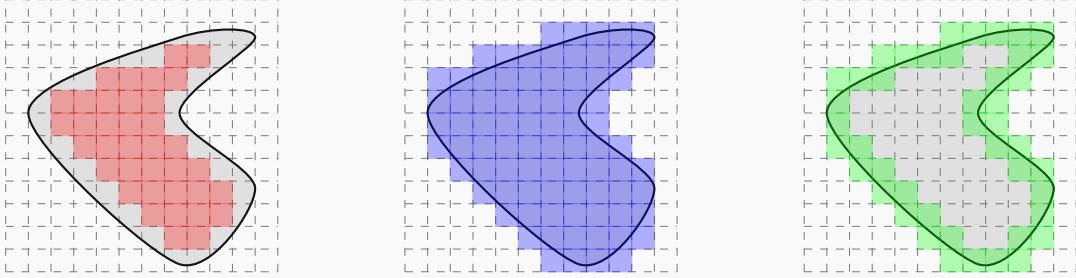


FIGURE 1.3. Simple sets approximating the inner (red) and outer Jordan content (blue) of a region in dimension $d = 2$. With the red boxes removed from the blue, we get a simple set covering the boundary (in green).

THEOREM 1.6

Let $E \subseteq \mathbb{R}^d$ be a bounded set. The following are equivalent:

- (i) E is Jordan measurable;
- (ii) $J_*(E) = J^*(E)$ (in which case $J(E)$ is equal to this same value);
- (iii) $J^*(\partial E) = 0$.

PROOF. We will prove the $d = 1$ case. The multidimensional case is similar but more notationally cumbersome, so we omit it to avoid additional technical details that would largely obscure the main ideas.

(i) $\iff$ (ii). To establish this equivalence, it suffices to show

$$U_B(\mathbb{1}_E) = J^*(E) \quad \text{and} \quad L_B(\mathbb{1}_E) = J_*(E)$$

for any box (interval) $B \supseteq E$. Let us prove $U_B(\mathbb{1}_E) = J^*(E)$.

CLAIM 1. $U_B(\mathbb{1}_E) \leq J^*(E)$.

Let $\varepsilon > 0$. Then from the definition of the outer Jordan content, there exists a simple set $S \subseteq \mathbb{R}$ such that $E \subseteq S$ and $\text{Vol}(S) < J^*(E) + \varepsilon$. By assumption, B is an interval containing E , so $S \cap B$ is also a simple set containing E , and $\text{Vol}(S \cap B) \leq \text{Vol}(S) < J^*(E) + \varepsilon$. We may therefore assume without loss of generality that $S \subseteq B$. Write $B = [a, b]$ and $S = [a_1, b_1] \sqcup [a_2, b_2] \sqcup \cdots \sqcup [a_n, b_n]$ with $a \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n \leq b$. We define a Darboux partition^a P of $[a, b]$ by $P = (x_i)_{i=0}^{2n+1}$ with $x_0 = a$, $x_1 = a_1$, $x_2 = b_1$, $\dots$, $x_{2n-1} = a_n$, $x_{2n} = b_n$, $x_{2n+1} = b$. Then since $E \subseteq S$, we have

$$\begin{aligned} U_B(\mathbb{1}_E, P) &= \sum_{i=1}^{2n+1} \sup_{x_{i-1} \leq x \leq x_i} \mathbb{1}_E(x) \cdot (x_i - x_{i-1}) \\ &\leq 0 \cdot (a_1 - a) + 1 \cdot (b_1 - a_1) + 0 \cdot (a_2 - b_1) + \cdots + 1 \cdot (b_n - a_n) + 0 \cdot (b - b_n) \\ &= \text{Vol}(S). \end{aligned}$$

Hence, $U_B(\mathbb{1}_E) \leq U_B(\mathbb{1}_E, P) \leq \text{Vol}(S) < J^*(E) + \varepsilon$. This proves the claim.

^aStrictly speaking, this may fail to be a Darboux partition, since some of the points are allowed to coincide. However, the value we compute for $U_B(\mathbb{1}_E, P)$ will be the correct value for the partition where we remove repetitions of the same point.

CLAIM 2. $J^*(E) \leq U_B(\mathbb{1}_E)$.

Let $\varepsilon > 0$. Write $B = [a, b]$. Then there exists a Darboux partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $U_B(\mathbb{1}_E, P) < U_B(\mathbb{1}_E) + \varepsilon$. Let $M_i = \sup_{x_{i-1} \leq x \leq x_i} \mathbb{1}_E(x) \in \{0, 1\}$, and note that, by definition, $U_B(\mathbb{1}_E, P) = \sum_{i=1}^n M_i(x_i - x_{i-1})$. Let $I \subseteq \{1, \dots, n\}$ be the set $I = \{1 \leq i \leq n : M_i = 1\}$, and let $S = \bigcup_{i \in I} [x_{i-1}, x_i]$. Then S is a simple set with length $\text{Vol}(S) = \sum_{i \in I} (x_i - x_{i-1}) = U_B(\mathbb{1}_E, P)$. Moreover, $E \subseteq S$, since S is the union of all intervals that have nonempty intersection with E . Thus, $J^*(E) \leq \text{Vol}(S) = U_B(\mathbb{1}_E, P) < U_B(\mathbb{1}_E) + \varepsilon$.

The identity $L_B(\mathbb{1}_E) = J_*(E)$ is proved similarly.

(ii) $\iff$ (iii). It suffices to prove $J^*(\partial E) = J^*(E) - J_*(E)$. (See Figure 1.3.)

CLAIM 3. $J^*(\partial E) \leq J^*(E) - J_*(E)$.

Let $\varepsilon > 0$. Let S_1 be a simple set such that $E \subseteq S_1$ and $\text{Vol}(S_1) < J^*(E) + \frac{\varepsilon}{2}$. Since S_1 is closed, we have $\overline{E} \subseteq S_1$. Let S_2 be a simple set with $S_2 \subseteq E$ such that $\text{Vol}(S_2) > J_*(E) - \frac{\varepsilon}{2}$. Note that $\text{int}(S_2) \subseteq \text{int}(E)$. Therefore, $S = S_1 \setminus \text{int}(S_2)$ is a simple set and $\partial E = \overline{E} \setminus \text{int}(E) \subseteq S$, so $J^*(\partial E) \leq \text{Vol}(S) = \text{Vol}(S_2) - \text{Vol}(S_1) < J^*(E) - J_*(E) + \varepsilon$. But ε was arbitrary, so we conclude $J^*(\partial E) \leq J^*(E) - J_*(E)$.

CLAIM 4. $J^*(E) - J_*(E) \leq J^*(\partial E)$.

Let $\varepsilon > 0$, and let $S \supseteq \partial E$ be a simple set with $\text{Vol}(S) < J^*(\partial E) + \frac{\varepsilon}{2}$. Write $S = \bigsqcup_{i=1}^n [a_i, b_i]$ with $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_n \leq b_n$. Let $[a, b] \subseteq \mathbb{R}$ such that $E \subseteq [a, b]$ and $a < a_1$ and $b < b_n$. For notational convenience, let $b_0 = a$ and $a_{n+1} = b$. Let $I \subseteq \{0, \dots, n\}$ be the collection of indices i such that $(b_i, a_{i+1}) \cap E \neq \emptyset$. For each $i \in I$, we claim that $(b_i, a_{i+1}) \subseteq E$. If not, then (b_i, a_{i+1}) contains a boundary point of E , but $\partial E \subseteq S$, so this is a contradiction. Thus, $S' = \bigcup_{i \in I} [b_i, a_{i+1}]$ is a simple set with $\text{int}(S') \subseteq E$. Shrinking slightly each interval in S' , we obtain a simple set

$$S'' = \bigcup_{i \in I} \left[b_i + \frac{\varepsilon}{4(n+1)}, a_{i+1} - \frac{\varepsilon}{4(n+1)} \right]$$

such that $S'' \subseteq E$. Moreover, $\text{Vol}(S'') \geq \text{Vol}(S') - \frac{\varepsilon}{2(n+1)}|I| \geq \text{Vol}(S') - \frac{\varepsilon}{2}$. Noting that $S \cup S'$ is a simple set containing E , we arrive at the inequality

$$J^*(E) - J_*(E) \leq \text{Vol}(S \cup S') - \text{Vol}(S'') = \text{Vol}(S) + \text{Vol}(S') - \text{Vol}(S'') < J^*(\partial E) + \varepsilon.$$

This completes the proof of Theorem 1.6. □

EXAMPLE 1.7

The sets $\mathbb{Q} \cap [0, 1]$ and $[0, 1] \setminus \mathbb{Q}$ are not Jordan measurable (see Problem 1.1).

In addition to the above example, there are many other “nice” sets that are not Jordan measurable. There are, for instance, bounded open sets in $\mathbb{R}$ that are not Jordan measurable. We will work out one such example in detail.

EXAMPLE 1.8

The complement U of the fat Cantor set (also known as the Smith–Volterra–Cantor set) $K \subseteq [0, 1]$ is Jordan non-measurable. We construct K iteratively, starting from $[0, 1]$, by removing intervals of length 4^{-n} at step n . In other words, at step n , we remove an interval of length 4^{-n} around each rational point with denominator 2^n .

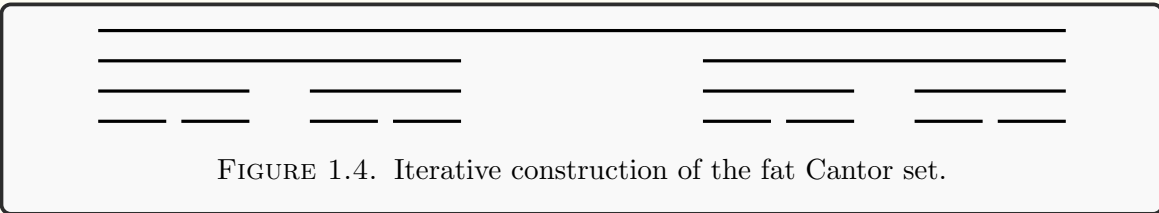


FIGURE 1.4. Iterative construction of the fat Cantor set.

Let

$$U = \bigcup_{n=0}^{\infty} \bigcup_{j=1}^{2^n} \left(\frac{2j+1}{2^{n+1}} - \frac{1}{2 \cdot 4^{n+1}}, \frac{2j+1}{2^{n+1}} + \frac{1}{2 \cdot 4^{n+1}} \right).$$

Then $K = [0, 1] \setminus U$. The inner Jordan content of U is

$$J_*(U) = \sum_{n=0}^{\infty} \sum_{j=1}^{2^n} \text{Len} \left(\frac{2j+1}{2^{n+1}} - \frac{1}{2 \cdot 4^{n+1}}, \frac{2j+1}{2^{n+1}} + \frac{1}{2 \cdot 4^{n+1}} \right) = \sum_{n=0}^{\infty} 2^n \cdot \frac{1}{4^{n+1}} = \frac{1}{4} \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{2}.$$

However, $\overline{U} = [0, 1]$ (since U contains every rational number whose denominator is a power of 2), so the outer Jordan content of U is $J^*(U) = J^*([0, 1]) = 1$.

3. Limits of Integrable Functions

You may recall from the theory of Riemann integration that *uniform* limits of Riemann integrable functions are Riemann integrable, and one may in this case interchange the order of taking limits and computing the integral. More precisely:

THEOREM 1.9

Let B be a box in $\mathbb{R}^d$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of Riemann integrable functions on B , and suppose f_n converges uniformly to a function $f : B \rightarrow \mathbb{R}$. Then f is Riemann integrable, and

$$\int_B f(\mathbf{x}) \, d\mathbf{x} = \lim_{n \rightarrow \infty} \int_B f_n(\mathbf{x}) \, d\mathbf{x}.$$

One of the deficiencies of the Riemann–Darboux–Jordan approach to integration and measurement is that pointwise (non-uniform) limits do not share this property.

EXAMPLE 1.10

Enumerate the set $\mathbb{Q} \cap [0, 1] = \{q_1, q_2, \dots\}$. Let $f_n : [0, 1] \rightarrow [0, 1]$ be the function

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{q_1, \dots, q_n\} \\ 0, & \text{otherwise.} \end{cases}$$

Then f_n is Riemann integrable and $f_n \rightarrow \mathbb{1}_{\mathbb{Q} \cap [0, 1]}$ pointwise, but $\mathbb{1}_{\mathbb{Q} \cap [0, 1]}$ is not Riemann integrable.

Since analysis so often deals with limits, it is desirable to develop a theory of integration that accommodates pointwise limits. The Lebesgue measure and Lebesgue integral resolve this shortcoming.

4. The Solution of Lebesgue

The Jordan non-measurable set in Example 1.8 appears to have a sensible notion of “length.” Indeed, the complement U , being a disjoint union of intervals, could be reasonably assigned as a “length” the sum of the lengths of the (countably many) intervals of which it is made. This produces a value of $\frac{1}{2}$ for the length of U , and so we should take K to also have length $\frac{1}{2}$, since $K \sqcup U = [0, 1]$ is an interval of length 1. The feature that U is a disjoint union of intervals turns out to not be any special feature of U at all but instead a general feature of open sets in $\mathbb{R}$.

PROPOSITION 1.11

Let $U \subseteq \mathbb{R}$ be an open set. Then U can be expressed as a countable disjoint union of open intervals.

PROOF. Problem 1.2. □

By Proposition 1.11, it seems reasonable to define the length of an open set $U \subseteq \mathbb{R}$ as follows. Write $U = (a_1, b_1) \sqcup (a_2, b_2) \sqcup \dots$ as a disjoint union of open intervals, and define its length as

$(b_1 - a_1) + (b_2 - a_2) + \dots$. Then open sets may play the role that simple sets played in the definition of the Jordan content, and this leads to the Lebesgue measure.

REMARK. In higher dimensions, Proposition 1.11 needs to be modified, but one can still reasonably talk about the d -dimensional volume of open sets in $\mathbb{R}^d$. See Problems 1.3 and 1.6.

DEFINITION 1.12

Let $E \subseteq \mathbb{R}^d$.

- The *outer Lebesgue measure of E* is the quantity

$$\lambda^*(E) = \inf \{ \text{Vol}(U) : U \supseteq E \text{ is open} \}$$

$$= \inf \left\{ \sum_{j=1}^{\infty} \text{Vol}(B_j) : B_1, B_2, \dots \text{ are boxes, and } E \subseteq \bigcup_{j=1}^{\infty} B_j \right\}.$$

- The set E is *Lebesgue measurable* (with *Lebesgue measure* $\lambda(E) = \lambda^*(E)$) if for every $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}^d$ such that $E \subseteq U$ and $\lambda^*(U \setminus E) < \varepsilon$.

PROPOSITION 1.13

If $E \subseteq \mathbb{R}^d$ is Jordan measurable, then E is Lebesgue measurable and $J(E) = \lambda(E)$.

The family of Lebesgue measurable sets is much larger than the family of Jordan measurable sets. Among the several nice properties of the Lebesgue measure (and abstract measures) that we will see later in the course are:

PROPOSITION 1.14

- (1) If $(E_n)_{n \in \mathbb{N}}$ are Lebesgue measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ and $\bigcap_{n=1}^{\infty} E_n$ are Lebesgue measurable.
- (2) If $(E_n)_{n \in \mathbb{N}}$ are pairwise disjoint and Lebesgue measurable, then $\lambda(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda(E_n)$.
- (3) If $E_1 \subseteq E_2 \subseteq \dots \subseteq \mathbb{R}^d$ are Lebesgue measurable sets, then $\lambda(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} \lambda(E_n)$.
- (4) If $E_1 \supseteq E_2 \supseteq \dots$ are Lebesgue measurable subsets of $\mathbb{R}^d$ and $\lambda(E_1) < \infty$, then $\lambda(\bigcap_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} \lambda(E_n)$.

5. Applications of Abstract Measure Theory

The mathematical language and tools encompassed in measure theory play a foundational role in many other areas of mathematics. A highly abbreviated sampling follows.

PROBABILITY THEORY. Measure theory provides the axiomatic foundations of probability theory, providing rigorous notions of *random variables* and *probabilities of events*. Important limit laws (the law of large numbers and central limit theorem, for example) are phrased mathematically using measure-theoretic notions of convergence.

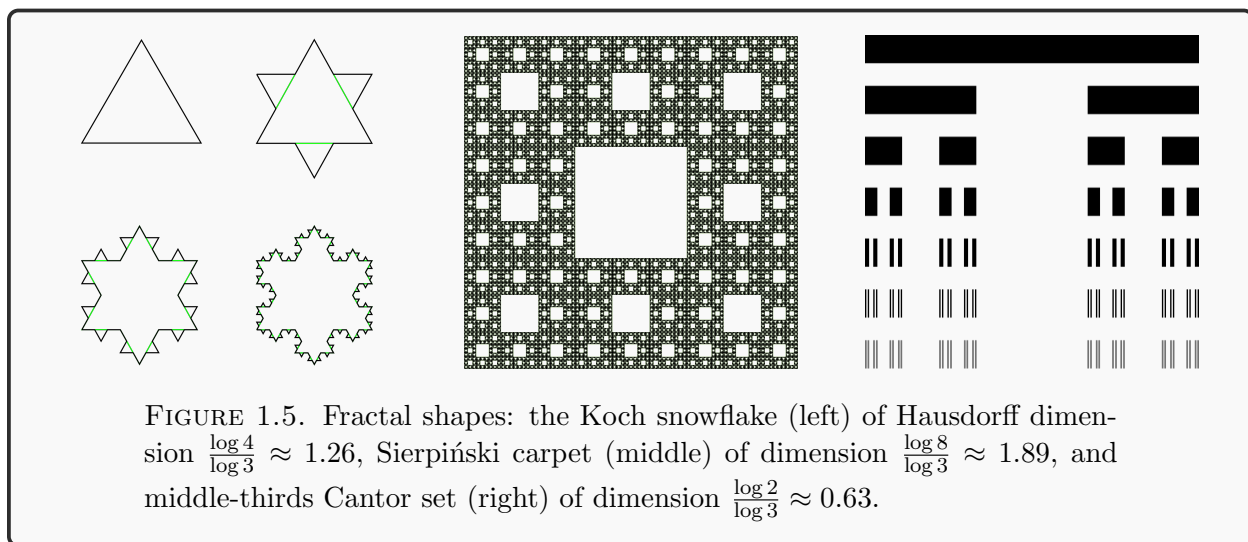
FOURIER ANALYSIS. Periodic (say, continuous or Riemann-integrable) functions on the real line have corresponding Fourier series representations $f(x) \sim \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$. The functions $e^{2\pi i n x}$ are orthonormal, and Parseval's identity gives $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 = \int_0^1 |f(x)|^2 dx$. Given a sequence $(a_n)_{n \in \mathbb{N}}$, one may ask whether $\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$ is the Fourier expansion of some function f , and if

so, what properties does f have? Another natural question is whether the series $\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$ actually converges to the function f , and if so, in which sense? Both of these questions are properly answered in a measure-theoretic framework. If one is interested in decomposing functions defined on other groups (for instance, on compact abelian groups) into their Fourier series, then one also needs to develop a method of integrating functions on groups in order to compute Fourier coefficients and make sense of Parseval's identity.

FUNCTIONAL ANALYSIS AND OPERATOR THEORY. When one studies familiar concepts from linear algebra in infinite-dimensional spaces, measures become unavoidable for many tasks. For example, versions of the spectral theorem (generalizing the representation of suitable matrices in terms of their eigenvalues and eigenvectors) for operators on infinite-dimensional spaces require the abstract notion of a measure.

ERGODIC THEORY. Ergodic theory was developed to study the long-term statistical behavior of dynamical (time-dependent) systems, providing a framework to resolve important problems in physics related to the “ergodic hypothesis” in thermodynamics and the “stability” of the solar system. It turns out that the appropriate mathematical formalism for understanding these problems comes from abstract measure theory.

FRactal GEOMETRY. Self-similar geometric objects such as the Koch snowflake, Sierpiński carpet, and the middle-thirds Cantor set (see Figure 1.5) can be meaningfully assigned a notion of “dimension” that can take a non-integer value. How does one determine the dimension of a fractal object? There are several different approaches to dimension, but one of the most popular is the *Hausdorff dimension*, which relies on a family of measures that interpolate between the integer-dimensional Lebesgue measures.



Additional Reading

This introductory chapter is heavily influenced by the book of Tao [5] on measure theory. Many of the results in this chapter are discussed in greater detail in [5, Section 1.1].

Exercises

PROBLEM 1.1. Show that $J^*(\mathbb{Q} \cap [0, 1]) = J^*([0, 1] \setminus \mathbb{Q}) = 1$, and $J_*(\mathbb{Q} \cap [0, 1]) = J_*([0, 1] \setminus \mathbb{Q}) = 0$.

PROBLEM 1.2. Let $U \subseteq \mathbb{R}$ be an open set. Show that U can be written as a disjoint union of countably many open intervals.

PROBLEM 1.3. Let $U = \{(x, y) : x^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ be the open unit disk. Show that U cannot be expressed as a disjoint union of countably many open boxes.

PROBLEM 1.4. Give an example to show that the statement

$$\lambda^*(E) = \sup_{U \subseteq E, U \text{ open}} \lambda^*(U).$$

is false.

PROBLEM 1.5 (Area Interpretation of the Riemann Integral). Let $[a, b]$ be an interval and $f : [a, b] \rightarrow [0, \infty)$ a bounded function. Show that f is Riemann integrable if and only if the set

$$E_+ = \{(x, t) : a \leq x \leq b, 0 \leq t \leq f(x)\}$$

is a Jordan measurable set in $\mathbb{R}^2$, in which case

$$\int_a^b f(x) \, dx = J(E_+).$$

PROBLEM 1.6. Let $U \subseteq \mathbb{R}^d$ be an open set. Show that U can be written as a disjoint union of countably many half-open boxes (i.e., sets of the form $B = \prod_{i=1}^d [a_i, b_i)$).

CHAPTER 2

Measure Spaces

1. σ -Algebras

Before defining measures, we must determine which subsets of a given set X we would like to be able to measure. The full set X should be measurable, and we should allow ourselves to perform the basic set-theoretic operations (complements, unions, and intersections). Allowing *finite* unions and intersections produces an *algebra* of sets. Algebras are a very useful notion, but (as with the Jordan content discussed in the previous chapter) they are insufficient for appropriately handling limits. We will therefore upgrade from algebras to σ -algebras:

DEFINITION 2.1

Let X be a set. A σ -algebra on X is a family $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X with the following properties:

- $X \in \mathcal{B}$;
- If $B \in \mathcal{B}$, then $X \setminus B \in \mathcal{B}$;
- If $(B_n)_{n \in \mathbb{N}}$ is a countable family of elements of $\mathcal{B}$, then $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

REMARK. In the definition of a σ -algebra, we have made no explicit mention of intersections. However, by De Morgan's laws, we can also generate the countable intersection of sets: $\bigcap_{n \in \mathbb{N}} B_n = X \setminus (\bigcup_{n \in \mathbb{N}} (X \setminus B_n))$.

EXAMPLE 2.2

Some examples of σ -algebras include the following:

- For any set X , the power set $\mathcal{P}(X)$ is a σ -algebra, as is the pair $\{\emptyset, X\}$.
- The family $\mathcal{B} = \{B \subseteq \mathbb{R} : \text{either } B \text{ or } \mathbb{R} \setminus B \text{ is countable}\}$ of countable and co-countable subsets of $\mathbb{R}$ is a σ -algebra.
- Unions of unit-length intervals in $\mathbb{R}$ form a σ -algebra $\mathcal{B} = \{\bigcup_{n \in S} [n, n+1) : S \subseteq \mathbb{Z}\}$.

PROPOSITION 2.3

Suppose $(\mathcal{B}_i)_{i \in I}$ is a family of σ -algebras on X . Then $\bigcap_{i \in I} \mathcal{B}_i$ is a σ -algebra.

PROOF. Let $\mathcal{B} = \bigcap_{i \in I} \mathcal{B}_i$.

For every $i \in I$, we have $X \in \mathcal{B}_i$, so $X \in \mathcal{B}$.

Suppose $B \in \mathcal{B}$. Then $B \in \mathcal{B}_i$ for every $i \in I$, so $X \setminus B \in \mathcal{B}_i$ for every $i \in I$. Hence, $X \setminus B \in \mathcal{B}$.

Let $(B_n)_{n \in \mathbb{N}}$ be a countable family of sets in $\mathcal{B}$. For each $i \in I$, the sets $(B_n)_{n \in \mathbb{N}}$ belong to $\mathcal{B}_i$, so $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}_i$. Therefore, $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$. □

DEFINITION 2.4

The *σ -algebra generated by a family $\mathcal{S} \subseteq \mathcal{P}(X)$* is the smallest σ -algebra containing $\mathcal{S}$, denoted by $\sigma(\mathcal{S})$.

REMARK. Note that $\sigma(\mathcal{S})$ is well-defined by Proposition 2.3:

$$\sigma(\mathcal{S}) = \bigcap \{ \mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{-algebra on } X, \mathcal{S} \subseteq \mathcal{B} \}.$$

In topological spaces (such as the real line), we will often consider the σ -algebra generated by the topology.

DEFINITION 2.5

Let (X, τ) be a topological space. The *Borel σ -algebra* is the σ -algebra generated by the open subsets of X , i.e. $\text{Borel}(X) = \sigma(\tau)$.

Borel sets can be placed in a hierarchy in terms of their level of complexity. At the simplest level are the open (G) and closed (F) sets. Next come countable intersections of open sets (G_δ sets) and countable unions of closed sets (F_σ sets) and so on.

$$\begin{array}{cccccccccccc} \Sigma_1^0 = G & \xrightarrow{\cap} & \Pi_2^0 = G_\delta & \xrightarrow{\cup} & \Sigma_3^0 = G_{\delta\sigma} & \xrightarrow{\cap} & \Pi_4^0 = G_{\delta\sigma\delta} & \xrightarrow{\cup} & \dots & \xrightarrow{\cup} & \Sigma_\omega^0 & \xrightarrow{\cap} & \dots \\ \updownarrow c & & \updownarrow c & & \updownarrow c & & \updownarrow c & & & & \updownarrow c & & \\ \Pi_1^0 = F & \xrightarrow{\cup} & \Sigma_2^0 = F_\sigma & \xrightarrow{\cap} & \Pi_3^0 = F_{\sigma\delta} & \xrightarrow{\cup} & \Sigma_4^0 = F_{\sigma\delta\sigma} & \xrightarrow{\cap} & \dots & \xrightarrow{\cap} & \Pi_\omega^0 & \xrightarrow{\cup} & \dots \end{array}$$

FIGURE 2.1. The Borel hierarchy for subsets of a topological space.

The placement of a (Borel) set within the Borel hierarchy is a useful notion of “complexity” for sets. Intuitively speaking, if a set is lower down in the Borel hierarchy, then it is in some sense easier to define than a set higher up the hierarchy. Determining where sets occur in the Borel hierarchy (or if they are Borel at all) is a common theme in an area of mathematical logic known as *descriptive set theory*. We will largely not concern ourselves with such problems in this course, but some suggested additional reading appears at the end of this chapter for those who are interested.

In our development of the abstract theory of measures (where we may not even have a topology to work with), our object of study will be arbitrary sets X equipped with a σ -algebra.

DEFINITION 2.6

A *measurable space* is a pair $(X, \mathcal{B})$, where X is a set and $\mathcal{B}$ is a σ -algebra on X . Elements of the σ -algebra $\mathcal{B}$ are called *measurable sets*.

2. Measurable Functions

Recall that a function $f : X \rightarrow Y$ from one topological space to another is continuous if the preimage of every open set in Y is open in X . Measurable functions are defined analogously, but with “open” replaced by “measurable.”

DEFINITION 2.7

Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces. A function $f : X \rightarrow Y$ is *measurable* if for every $C \in \mathcal{C}$, one has $f^{-1}(C) \in \mathcal{B}$.

Some basic properties of measurable functions that will be used frequently are as follows:

PROPOSITION 2.8

- (1) Let $(X, \mathcal{B})$, $(Y, \mathcal{C})$, and $(Z, \mathcal{D})$ be measurable spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be measurable functions. Then $g \circ f : X \rightarrow Z$ is measurable.
- (2) Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces, and let $f : X \rightarrow Y$. Suppose $\mathcal{S} \subseteq \mathcal{P}(Y)$ is a family of sets such that $\sigma(\mathcal{S}) = \mathcal{C}$. If $f^{-1}(S) \in \mathcal{B}$ for every $S \in \mathcal{S}$, then f is a measurable function.
- (3) Suppose X and Y are topological spaces and $\mathcal{B} = \text{Borel}(X)$ and $\mathcal{C} = \text{Borel}(Y)$ are the Borel σ -algebras on X and Y respectively. Then every continuous function $f : X \rightarrow Y$ is measurable.

PROOF. (1) Let $D \in \mathcal{D}$. Since g is measurable, we have $C = g^{-1}(D) \in \mathcal{C}$. Then since f is measurable, $B = f^{-1}(C) \in \mathcal{B}$. But $B = f^{-1}(g^{-1}(D)) = (g \circ f)^{-1}(D)$, so $g \circ f$ is measurable.

(2) Let $\mathcal{F} = \{E \subseteq Y : f^{-1}(E) \in \mathcal{B}\}$. We claim that $\mathcal{F}$ is a σ -algebra. Then since $\mathcal{S} \subseteq \mathcal{F}$, we conclude that $\mathcal{C} = \sigma(\mathcal{S}) \subseteq \mathcal{F}$, so f is measurable. Let us now prove the claim:

- $f^{-1}(Y) = X \in \mathcal{B}$, so $Y \in \mathcal{F}$.
- Suppose $E \in \mathcal{F}$. Then $f^{-1}(Y \setminus E) = X \setminus \underbrace{f^{-1}(E)}_{\in \mathcal{B}} \in \mathcal{B}$, so $Y \setminus E \in \mathcal{F}$.
- Suppose $E_1, E_2, \dots \in \mathcal{F}$, and let $E = \bigcup_{n \in \mathbb{N}} \underbrace{E_n}_{\in \mathcal{B}}$. Then

$$f^{-1}(E) = \bigcup_{n \in \mathbb{N}} \underbrace{f^{-1}(E_n)}_{\in \mathcal{B}} \in \mathcal{B},$$

so $E \in \mathcal{F}$.

This proves that $\mathcal{F}$ is a σ -algebra on Y .

(3) follows from (1) by taking $\mathcal{S}$ to be the collection of open sets in Y . □

3. The Extended Real Numbers and Extended Real-Valued Functions

One obtains an important class of measurable functions when one considers functions defined on a measurable space taking real values. For many applications and in order to account more fully for limits of functions, it is often convenient to work with the slightly more general concept of *extended* real-valued functions.

DEFINITION 2.9

The *extended real numbers* are the set $[-\infty, \infty] = \mathbb{R} \cup \{\infty, -\infty\}$ with the following topological and algebraic properties:

- The topology on $[-\infty, \infty]$ is generated by open intervals (a, b) with $a, b \in \mathbb{R}$ and sets of the form $(a, \infty] = (a, \infty) \cup \{\infty\}$ and $[-\infty, b) = (-\infty, b) \cup \{-\infty\}$ for $a, b \in \mathbb{R}$.

- Addition is extended as a commutative operation with $\infty + x = \infty$ and $-\infty + x = -\infty$ for real numbers $x \in \mathbb{R}$. For addition of two infinite quantities, we define $\infty + \infty = \infty$ and $-\infty + (-\infty) = -\infty$. However, $-\infty + \infty$ is undefined.
- Multiplication is also extended as a commutative operation with the properties

$$\begin{aligned} x \in (0, \infty) &\implies \infty \cdot x = \infty \quad \text{and} \quad -\infty \cdot x = -\infty; \\ x \in (-\infty, 0) &\implies \infty \cdot x = -\infty \quad \text{and} \quad -\infty \cdot x = \infty. \end{aligned}$$

By convention, we define $\infty \cdot 0 = -\infty \cdot 0 = 0$. Multiplication of infinities is defined by $\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$, and $-\infty \cdot \infty = -\infty$.

The topology we have defined on $[-\infty, \infty]$ is the *two-point compactification* of $\mathbb{R}$. You will check in the exercises (Problem 2.1) that $[-\infty, \infty]$ is indeed a compact space (that is homeomorphic to a closed interval, say $[0, 1]$). The algebraic operations on $[-\infty, \infty]$ are all as one would expect, with one exception: $\infty \cdot 0$ is often considered as an “indeterminate form”, but here we have given it a definite value of 0. The reason for this convention is the following proposition, which you will also prove in the exercises:

PROPOSITION 2.10

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $[-\infty, \infty]$, and let $c \in \mathbb{R}$. If $(x_n)_{n \in \mathbb{N}}$ converges to an extended real number, then the sequence $(cx_n)_{n \in \mathbb{N}}$ also converges, and

$$\lim_{n \rightarrow \infty} (cx_n) = c \cdot \lim_{n \rightarrow \infty} x_n. \quad (2.1)$$

PROOF. Problem 2.2. □

In order to have the desirable property (2.1), one has no choice but to define $\infty \cdot 0 = 0$: by taking the sequence $x_n = n$, we have

$$0 \cdot \infty = 0 \cdot \lim_{n \rightarrow \infty} n = \lim_{n \rightarrow \infty} (0 \cdot n) = 0.$$

WARNING: Property (2.1) does not hold for $c \in \{\infty, -\infty\}$, as can be seen by taking a sequence $(x_n)_{n \in \mathbb{N}}$ that converges to 0.

We say that an extended real-valued function $f : X \rightarrow [-\infty, \infty]$ defined on a measurable space $(X, \mathcal{B})$ is $\mathcal{B}$ -*measurable* (or simply *measurable*) if it is measurable as a function between the measurable spaces $(X, \mathcal{B})$ and $([-\infty, \infty], \text{Borel}([-\infty, \infty]))$. Since we will always take the same σ -algebra on $[-\infty, \infty]$, we omit explicit reference to the Borel σ -algebra when discussing measurable extended real-valued functions.

PROPOSITION 2.11

Let $(X, \mathcal{B})$ be a measurable space.

- (1) Let $f : X \rightarrow [-\infty, \infty]$. The following are equivalent:
 - (a) f is measurable;
 - (b) for every $c \in \mathbb{R}$, $f^{-1}((c, \infty]) \in \mathcal{B}$;
 - (c) for every $c \in \mathbb{R}$, $f^{-1}([c, \infty]) \in \mathcal{B}$;
 - (d) for every $c \in \mathbb{R}$, $f^{-1}([-\infty, c)) \in \mathcal{B}$;
 - (e) for every $c \in \mathbb{R}$, $f^{-1}([-\infty, c]) \in \mathcal{B}$.

- (2) Suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence of measurable functions from X to $[-\infty, \infty]$. The following functions are also measurable:
- (a) $\sup_{n \in \mathbb{N}} f_n$;
 - (b) $\inf_{n \in \mathbb{N}} f_n$;
 - (c) $\limsup_{n \rightarrow \infty} f_n$;
 - (d) $\liminf_{n \rightarrow \infty} f_n$.
- (3) Suppose $f, g : X \rightarrow \mathbb{R}$ are measurable functions. Then $f + g$ and $f \cdot g$ are measurable.

NOTATION. For convenience, we will often write sets of the form $f^{-1}((c, \infty])$ as $\{f > c\}$ and similarly for $\{f \geq c\}$, $\{f < c\}$, and $\{f \leq c\}$.

PROOF OF PROPOSITION 2.11. (1) By Proposition 2.8(2), it suffices to check that each of the relevant collections of intervals generates the Borel σ -algebra on $[-\infty, \infty]$. Let us show that the collection of intervals $(c, \infty]$ for $c \in \mathbb{R}$ generates the Borel σ -algebra. All of the other proofs are similar, so we omit them.

Let $\mathcal{S} = \{(c, \infty] : c \in \mathbb{R}\}$. Note that every element of $\mathcal{S}$ is open in $[-\infty, \infty]$, so $\sigma(\mathcal{S}) \subseteq \text{Borel}([-\infty, \infty])$. On the other hand, we can write $(a, b] = (a, \infty] \setminus (b, \infty]$ for $a, b \in \mathbb{R}, a < b$. Every open set in $\mathbb{R}$ is a countable (disjoint) union of such intervals, so every open subset of $\mathbb{R}$ is contained in $\sigma(\mathcal{S})$. We obtain the additional open sets in $[-\infty, \infty]$ from the rays $(c, \infty] \in \mathcal{S}$ and

$$[-\infty, c) = \bigcap_{n \in \mathbb{N}} \left[-\infty, c + \frac{1}{n} \right] = \bigcap_{n \in \mathbb{N}} \left([-\infty, \infty] \setminus \left(c + \frac{1}{n}, \infty \right] \right) \in \sigma(\mathcal{S}).$$

Thus, $\text{Borel}([-\infty, \infty]) \subseteq \sigma(\mathcal{S})$.

(2) We will use (1).

(a) Let $f = \sup_{n \in \mathbb{N}} f_n$. Note that $\{f > c\} = \bigcup_{n \in \mathbb{N}} \{f_n > c\}$. Each of the sets $\{f_n > c\}$ belongs to $\mathcal{B}$, so $\{f > c\} \in \mathcal{B}$.

(b) Similarly to (a), letting $f = \inf_{n \in \mathbb{N}} f_n$, we may express $\{f < c\} = \bigcup_{n \in \mathbb{N}} \underbrace{\{f_n < c\}}_{\in \mathcal{B}} \in \mathcal{B}$.

(c) Recall that $\limsup_{n \rightarrow \infty} f_n = \inf_{k \in \mathbb{N}} \sup_{n \geq k} f_n$, so measurability of $\limsup_{n \rightarrow \infty} f_n$ follows from (a) and (b).

(d) Similar to (c): $\liminf_{n \rightarrow \infty} f_n = \sup_{k \in \mathbb{N}} \inf_{n \geq k} f_n$.

(3) Let $A : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $M : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the maps $A(x, y) = x + y$ and $M(x, y) = xy$. Both of the maps A and M are continuous and therefore (Borel) measurable. Moreover, $(f+g)(x) = A(f(x), g(x))$ and $(f \cdot g)(x) = M(f(x), g(x))$. Since the composition of measurable maps is measurable (see Proposition 2.8(1)), it suffices to prove $h : x \mapsto (f(x), g(x))$ is a measurable function from X to $\mathbb{R}^2$. By Proposition 2.8(2), we only need to check preimages of sets generating the Borel σ -algebra on $\mathbb{R}^2$. For convenience, we will take the boxes $[a, b) \times [c, d)$ (the first homework problem was to show that every open set in $\mathbb{R}^2$ is a countable (disjoint) union of such boxes, so they generate the Borel σ -algebra). Observe that

$$h^{-1}([a, b) \times [c, d)) = f^{-1}([a, b)) \cap g^{-1}([c, d)) \in \mathcal{B},$$

since f and g are measurable, so h is indeed a measurable function. □

EXAMPLE 2.12

Let $(X, \mathcal{B})$ be a measurable space and $E \subseteq X$. The function $\mathbb{1}_E$ is measurable if and only if $E \in \mathcal{B}$.

4. Measures

We are now prepared to define measures on abstract measurable spaces.

DEFINITION 2.13

Let $(X, \mathcal{B})$ be a measurable space. A *measure* on $(X, \mathcal{B})$ is a function $\mu : \mathcal{B} \rightarrow [0, \infty]$ such that

- $\mu(\emptyset) = 0$;
- COUNTABLE ADDITIVITY: for any sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise disjoint elements of $\mathcal{B}$, one has $\mu(\bigsqcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \mu(E_n)$.

The triple $(X, \mathcal{B}, \mu)$ is called a *measure space*.

Nontrivial examples of measures take some effort to construct, and we will spend significant portions of the course discussing different methods for constructing interesting measures. However, there are a few immediate examples that do not require complicated constructions.

EXAMPLE 2.14

Examples of measures include:

- For any set X , the *counting measure* is a measure defined on the σ -algebra $\mathcal{P}(X)$ by $\mu(E) = |E|$ if E is a finite set and $\mu(E) = \infty$ if E is an infinite set.
- Given a point $x \in X$, the *Dirac measure* defined on $\mathcal{P}(X)$ is the measure $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ if $x \notin E$.

We will use the following basic properties of measures frequently throughout this course:

PROPOSITION 2.15

Let $(X, \mathcal{B}, \mu)$ be a measure space.

- (1) MONOTONICITY: For any $A, B \in \mathcal{B}$, if $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
- (2) COUNTABLE SUB-ADDITIVITY: For any sequence $(E_n)_{n \in \mathbb{N}}$ in $\mathcal{B}$,

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) \leq \sum_{n \in \mathbb{N}} \mu(E_n).$$

- (3) CONTINUITY FROM BELOW: If $E_1 \subseteq E_2 \subseteq \cdots \in \mathcal{B}$, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n).$$

- (4) CONTINUITY FROM ABOVE: If $E_1 \supseteq E_2 \supseteq \cdots \in \mathcal{B}$ and $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) = \lim_{n \rightarrow \infty} \mu(E_n).$$

PROOF. (1) Write $B = A \sqcup (B \setminus A)$. Then $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$, since μ takes nonnegative values.

(2) Define a new sequence of sets E'_n by $E'_1 = E_1$ and $E'_n = E_n \setminus \bigcup_{j=1}^{n-1} E_j$ for $n \geq 2$. Then the sets $(E'_n)_{n \in \mathbb{N}}$ are pairwise disjoint and satisfy $E'_n \subseteq E_n$ and $\bigsqcup_{n \in \mathbb{N}} E'_n = \bigcup_{n \in \mathbb{N}} E_n$. Therefore,

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \mu\left(\bigsqcup_{n \in \mathbb{N}} E'_n\right) = \sum_{n \in \mathbb{N}} \mu(E'_n) \leq \sum_{n \in \mathbb{N}} \mu(E_n),$$

where in the last step we have applied monotonicity of μ (property (1)).

(3) Let $E'_1 = E_1$ and $E'_n = E_n \setminus E_{n-1}$ for $n \geq 2$. For convenience, we will set $E_0 = \emptyset$ so that we also have $E'_1 = E_1 \setminus E_0$. Then

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \mu\left(\bigsqcup_{n \in \mathbb{N}} E'_n\right) = \sum_{n \in \mathbb{N}} \mu(E'_n) \stackrel{(*)}{=} \sum_{n \in \mathbb{N}} (\mu(E_n) - \mu(E_{n-1})) \stackrel{(**)}{=} \lim_{n \rightarrow \infty} \mu(E_n).$$

The step $(*)$ uses additivity of μ , and $(**)$ comes from the telescoping of the sum.

(4) Define a new sequence $A_n = E_1 \setminus E_n$. Then $\emptyset = A_1 \subseteq A_2 \subseteq \dots$, so

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n)$$

by (3). But $\bigcup_{n \in \mathbb{N}} A_n = E_1 \setminus \bigcap_{n \in \mathbb{N}} E_n$, so

$$\mu(E_1) - \mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) = \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n) = \mu(E_1) - \lim_{n \rightarrow \infty} \mu(E_n),$$

whence we deduce that (5) holds, since $\mu(E_1) < \infty$. □

EXAMPLE 2.16

Property (5) may fail if $\mu(E_1) = \infty$. Let $X = \mathbb{N}$, $\mathcal{B} = \mathcal{P}(\mathbb{N})$, and let μ be the counting measure. Let $E_n = \{m \in \mathbb{N} : m \geq n\}$. Then $\mu(E_n) = \infty$ for every $n \in \mathbb{N}$, but $\bigcap_{n \in \mathbb{N}} E_n = \emptyset$, so

$$\mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) = 0 \neq \infty = \lim_{n \rightarrow \infty} \mu(E_n).$$

Additional Reading

The content of this chapter is common to every text on abstract measure theory, though the order of presentation differs. We have elected to follow more or less the order of presentation from Rudin's *Real and Complex Analysis* [3, Chapter 1]. Alternative presentations can be found in [1, Sections 1.2, 1.3, and 2.1], and [5, Section 1.4].

Introductory texts on measure theory tend not to give much treatment to the Borel hierarchy or other topics in descriptive set theory (and we will also not expand on such topics within these lecture notes). Those interested in learning more can take a look at the book of Kechris [2] and/or the lecture notes of Tserunyan [6], which draw quite heavily on [2].

Exercises

PROBLEM 2.1. Prove that the extended real line $[-\infty, \infty]$ is homeomorphic to the closed unit interval $[0, 1]$.

PROBLEM 2.2. Prove Proposition 2.10.

PROBLEM 2.3. Let X, Y be sets and $f : X \rightarrow Y$ any function.

- (a) Prove that if $\mathcal{C} \subseteq \mathcal{P}(Y)$ is a σ -algebra on Y , then $\mathcal{B} = \{f^{-1}(C) : C \in \mathcal{C}\}$ is a σ -algebra on X .
- (b) Prove that for any family of sets $\mathcal{S} \subseteq \mathcal{P}(Y)$, we have $\sigma(f^{-1}(\mathcal{S})) = f^{-1}(\sigma(\mathcal{S}))$.

PROBLEM 2.4. Let $(X, \mathcal{B}, \mu)$ be a finite measure space, and let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra on X such that $\sigma(\mathcal{A}) = \mathcal{B}$. Show that for every $B \in \mathcal{B}$ and every $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(A \triangle B) < \varepsilon$.

PROBLEM 2.5. Let X be a set. A family of subsets $\mathcal{S} \subseteq \mathcal{P}(X)$ is a *semi-algebra* if

- $\emptyset, X \in \mathcal{S}$;
- if $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$;
- if $A, B \in \mathcal{S}$, then $A \setminus B = \bigsqcup_{i=1}^n C_i$ for some $C_1, \dots, C_n \in \mathcal{S}$.

Show that if $\mathcal{S}$ is a semi-algebra, then the algebra generated by $\mathcal{S}$ is

$$\mathcal{A}(\mathcal{S}) = \left\{ \bigcup_{i=1}^n A_i : n \in \mathbb{N}, A_i \in \mathcal{S} \right\}.$$

Can $\bigcup$ be replaced by $\bigsqcup$?

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